Equivalence of μ^p-Calculus and p-Automata

Extended Abstract

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ABSTRACT

An important characteristic of the modal μ-calculus is its strong connection with parity alternating tree automata. Here, we show that the probabilistic μ-calculus, μ^p-calculus, and p-automata (parity alternating Markov chain automata) have an equally strong connection. Namely, for every μ^p-calculus formula we can construct a p-automaton that accepts exactly those Markov chains that satisfy the formula. For every p-automaton we can construct a μ^p-calculus formula satisfied in exactly those Markov chains that are accepted by the automaton. The translation in one direction relies on a normal form of the calculus and in the other direction on the usage of vectorial μ^p-calculus.

KEYWORDS

μ-Calculus, Games, Automata, Probabilistic Temporal Logics

1 INTRODUCTION

The verification of probabilistic systems is an increasingly important area that has led to the development of new formalisms and tools for the evaluation of quantitative properties over stochastic models. The automata-theoretic approach to verification aims to reduce questions about specifications to questions about automata, defining a connection between logics and automata theory.

We focus on μ^p-calculus and p-automata. The former has been introduced in [1] as a probabilistic extension of Kozen's modal μ-calculus. The latter [3] are probabilistic alternating parity automata that read Markov chains as input. Acceptance of a Markov chain by a p-automaton is decided through an *obligation game*, that is, a turn-based stochastic parity game with obligations. We show that μ^p-calculus and p-automata have the same expressive power.

2 BACKGROUND

Markov Chains. A Markov chain $M = (S, s^{in}, L, P)$ over the set AP of atomic propositions is a probabilistic transition system. S is the set of locations; $s^{in} \in S$ is the initial location; L is a labelling function $L : S \to 2^{AP}$; and P is a stochastic matrix $P : S \times S \to [0, 1]$.

μ^p-Calculus. The μ^p-calculus allows the specification of properties bounded by a probability J (of the form $\{\geq, >\} \times [0, 1]$). This is done through the distinction between qualitative (Φ) and quantitative (Ψ) formulas, evaluated to values in the sets $\{0, 1\}$ and $[0, 1]$, respectively. A μ^p-calculus sentence is qualitative and might contain quantitative sub-formulas within the probabilistic operator $[\cdot]_J$. The syntax of μ^p-calculus is specified as follows:

$$\Phi ::= p \mid \neg p \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid [\Psi]_J \mid \mu X.\varphi \mid \nu X.\varphi$$
$$\Psi ::= \Phi \mid X \mid \psi_1 \wedge \psi_2 \mid \psi_1 \vee \psi_2 \mid \bigcirc\psi \mid \mu X.\psi \mid \nu X.\psi$$

The semantics of a μ^p-calculus sentence φ is given with respect to a Markov chain M and an interpretation ρ that associates a function $S \to [0, 1]$ with each variable X appearing in φ. Therefore, it is a mapping of type $[\![\varphi]\!]^{\rho}_{M} \to (S \to [0, 1])$. A Markov chain M satisfies φ, denoted $M \models \varphi$, if and only if the semantics evaluation of φ over the initial location s^{in} equals 1. The semantics of the calculus can also be defined in terms of *obligation games* [3].

p-Automata. We define the set $[\![Q]\!]_{>}$ as the set of states $[\![q]\!]_J$ that are bounded by a probability J; and denote by $B^+(X)$ the set of *positive boolean formulas* over elements x in the set X, that is, formulas built up from elements x combined with $true$, $false$, disjunctions, and conjunctions.

A p-automaton A over the set AP of atomic propositions is the tuple $A = (2^{AP}, Q, \varphi^{in}, \delta, \Omega)$, where 2^{AP} is the alphabet; Q is the set of states; $\varphi^{in} \in B^+([\![Q]\!]_{>})$ is the initial condition; $\delta : Q \times \Sigma \to B^+(Q \cup [\![Q]\!]_{>})$ is the transition function; and $\Omega : Q \to [0 \ldots k]$ is the parity acceptance condition. Acceptance of a Markov chain is decided through an *obligation game*; the set of Markov chains accepted by a p-automaton A is the language of A, denoted by $\mathcal{L}(A)$.

3 EQUIVALENCE

The translation from μ^p-calculus to p-automata relies on the formulas satisfying some syntactic requirements, defined as *well-formedness.* The conversion in the opposite direction exploits the *vectorial syntax* of the calculus. The proofs of both theorems use the game semantics of μ^p-calculus and p-automata to show that the proposed translations are correct.

Theorem 3.1 ([2]). *Let φ be a well-formed μ^p-calculus formula and A_φ the automaton resulting from its translation. Then, φ and A_φ are equivalent: the set of Markov chains that satisfy the formula φ corresponds to the language $\mathcal{L}(A_\varphi)$ recognised by the p-automaton A_φ. That is, $M \models \varphi$ iff $M \in \mathcal{L}(A_\varphi)$.*

Theorem 3.2 ([2]). *Let A be a p-automaton over the set AP of atomic propositions and $\vec{\varphi}_A$ the vectorial μ^p-calculus formula resulting from its conversion. Then, A and $\vec{\varphi}_A$ are equivalent: the set of Markov chains that constitute the language $\mathcal{L}(A)$ recognised by the p-automaton A coincides with the set of Markov chains that satisfy the vectorial formula $\vec{\varphi}_A$. That is, $M \in \mathcal{L}(A)$ iff $M \models \vec{\varphi}_A$.*

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