Tackling Number Theory analogues in Group Theory with GAP

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Keywords

GAP, perfect numbers, harmonic numbers, finite groups, quasi-Leinster groups

ACM Reference Format:

Iulia - Cătălina Pleşca, joint work with Marius Tărnăuceanu. 2025. Tackling Number Theory analogues in Group Theory with GAP. In . ACM, New York, NY, USA, 2 pages. https://doi.org/10.1145/nnnnnnn.nnnnnnn

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Abstract. In this work, we extend the parallelism between perfect numbers and Leinster groups introduced by Tom Leinster. We have two studies: finite groups with integer harmonic means of elements (analogues of harmonic numbers) and almost/quasi-Leinster groups which parallel almost and quasi-perfect numbers.

In the first part, we introduce the function “the harmonic mean of element orders of a finite group”, i.e.

$$h_m(G) = \frac{|G|}{m(G)}\,. \tag{1}$$

In our work, we put a dent in the following problem:

Question 1. Which are the finite groups G with $h_m(G) \in \mathbb{N}$?

In the second part, we investigate nilpotent almost/quasi-/Leinster groups and find some examples and conditions for the existence of such groups for classes of non-nilpotent groups: ZM (Zassenhaus metacyclic) groups, affine groups, using GAP.

1 Introduction

Throughout this paper, all the integers are positive and all the groups are finite. We denote the set of positive integers by $\mathbb{N}^*$. By *proper divisor* of an integer, we understand a divisor of the number that is different from the integer itself. Similarly, by a *proper subgroup* of a group, we understand a subgroup of the group except the group itself. For a group G, we denote by $L(G)$ the lattice of its subgroups and by $N(G)$ the sublattice of normal subgroups.

In 2001, Tom Leinster introduced a group theoretical analogue to perfect numbers: finite groups where the order of the group is equal to the sum of the orders of its proper normal subgroups. He called these groups perfect, but since this name was already used, these groups were later called Leinster groups.

Finding examples of Leinster groups of odd order proved to be difficult, taking ten years. An example was produced by François Brunault in reply to Tom Leinster's post on MathOverflow [5]. The use of computational programs brought thousands of examples of Leinster groups.

In the first part of our study, we give some immediate properties of the newly introduced function including a lower bound that is reached only by finite p-groups. We continue by characterizing the finite p-groups with integer h_m. In the end, we observe that $h_m^{-1}(2) = \{C_4, D_8\}$ and study $h_m^{-1}(3)$ with the help of GAP.

GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory. GAP provides a programming language, a library of thousands of functions implementing algebraic algorithms written in the GAP language as well as large data libraries of algebraic objects [3].

The second part aims to further study Leinster groups and introduce two new related concepts: almost and quasi-Leinster groups, group-theoretical analogues to almost and quasi-perfect numbers.

We introduce the necessary theoretical background and the new concepts. We also prove a result characterizing powers of even perfect numbers.

Afterwards, we characterize almost/quasi-/Leinster nilpotent groups. In the subsequent sections, we find examples of quasi-Leinster groups among ZM-groups and affine groups.

2 Integer harmonic mean of element orders

An **harmonic divisor number (Ore number)** is a positive integer $n \in \mathbb{N}$ for which the harmonic mean of their divisors is a positive integer: $H(n) = \frac{n\sigma_0(n)}{\sigma_1(n)} \in \mathbb{N}$.

Proposition 2.1 (O. Ore). *Every perfect number is a harmonic number.*

Conjecture (O. Ore). *There are no odd harmonic divisor numbers.*

A group is called **harmonic** if $H(G) \in \mathbb{N}$.

Theorem 2.2 (Baishya S.J., Das A.K.). *Every Leinster group of odd order is harmonic.*

Proposition 2.3 (I.C.P.,M.T.). *Let G be a finite group, $C(G) = \{H \le G \mid H$ cyclic$\}$ and p the smallest prime divisor of $|G|$. Then:*

$$h_m(G) \ge \frac{p|G|}{(p-1)|G|+1}.$$

Theorem 2.4 (I.C.P.,M.T.). *Let G be a finite p-group. Then $h_m(G) \in \mathbb{N}$ iff G is cyclic of order $p^{\sum_{i=1}^{s} p^i}$, with $s \in \mathbb{N}^*$ or $G \simeq D_8$.*

Corollary 2.5 (I.C.P.,M.T.). *The dihedral group D_8 is the only non-cyclic p-group with $h_m(G) \in \mathbb{N}$.*

Proposition 2.6 (I.C.P.,M.T.). *D_8 is the only dihedral group G with $h_m(G) \in \mathbb{N}$.*

Since $\min\{h_m(G) | G$ finite non-trivial$\} = \frac{4}{3}$, it follows there are no non-trivial groups with $h_m(G) = 1$.

Theorem 2.7 (I.C.P.,M.T.). *Let G be a finite group. Then $h_m(G) = 2$ iff $G \cong C_4$ or $G \cong D_8$.*

Conference'17, Washington, DC, USA

ACM ISBN 978-x-xxxx-xxxx-x/YYYY/MM
https://doi.org/10.1145/nnnnnnn.nnnnnnn

Theorem 2.8 (I.C.P.,M.T.). *There are no finite groups of odd order G with $h_m(G) = 3$.*

Example 2.1. We used GAP to look for finite groups G with $h_m(G) = 3$. The smallest example of such a group is $SmallGroup(12, 1)$.

Question 2 (Open). What is $\mathbb{N} \cap \mathrm{Im}(h_m)$?

Conjecture. *There are no groups G with $h_m(G) = 7$.*

3 A nice result

Definition 3.1. A **perfect number** is an integer that is equal to half the sum of its divisors. Alternatively, it is equal to the sum of its proper divisors.

Example 3.1. $6 = 1 + 2 + 3$, $28 = 1 + 2 + 4 + 7 + 14$.

Proposition 3.1 (Euclid). *Even perfect numbers are of the form $f(r) = 2^{r-1}(2^r - 1)$, where $r \geq 2$ and $2^r - 1$ is a Mersenne prime.*

Example 3.2 (The Great Internet Mersenne Prime Search). There are 52 known perfect even numbers: P_i, $i \in \{1, \ldots, 52\}$. The first seven are: $P_1 = 6 = f(2)$, $P_2 = 28 = f(3)$, $P_3 = 496 = f(5)$, $P_4 = 8128 = f(6)$, $P_5 = 33550336 = f(13)$, $P_6 = 8589869056 = f(17)$ and $P_7 = 137438691328 = f(19)$.

Corollary 3.2 (I.C.P., M.T.). *Let p be a prime, P_i an even perfect number and k a positive integer such that:*

$$P_i + 1 = p^k. \tag{2}$$

Then $k = 1$.

Example 3.3. Using GAP, we checked the first 39 even perfect numbers and found 4 solutions for (2):

$$P_i \in \{P_1 = 6, P_2 = 28, P_5 = 33550336, P_7 = 137438691328\}.$$

4 Quasi-Leinster groups

Given a finite group G, the following functions can be defined:

sum of orders of normal subgroups

$$\sigma(G) = \sum_{N \trianglelefteq G} |N| \tag{3}$$

sum of powers of orders

$$\sigma_z(G) = \sum_{N \trianglelefteq G} |N|^z, \text{ where } z \in \mathbb{C}. \tag{4}$$

average of orders of normal subgroups

$$\rho(G) = \frac{\sigma(G)}{|G|}. \tag{5}$$

- A finite group G is called
  - **Leinster** if $\rho(G) = 2$
  - **abundant** if $\rho(G) > 2$.
  - **deficient** if $\rho(G) < 2$.
- An abundant group is called **quasi-Leinster** if

$$\sigma(G) = 2|G| + 1. \tag{6}$$

- A deficient group is called **almost Leinster** if

$$\sigma(G) = 2|G| - 1. \tag{7}$$

Let G be a nilpotent group.

Proposition 4.1 (T. de Medts). $\rho(G) \leq 2 \iff$ *G cyclic and $|G|$ perfect or defficient.*

Proposition 4.2 (I.C.P., M.T.). *G is quasi-Leinster if and only if G is cyclic and $|G|$ is quasi-perfect.*

A **ZM-group** (Zassenhaus metacyclic) is a finite group with all Sylow subgroups cyclic.
The presentation of ZM-groups is given by:

$$ZM(m, n, r) = \langle a, b \mid a^m = b^n = 1, b^{-1}ab = a^r \rangle, \tag{8}$$

where the triplet $(m, n, r) \in \mathbb{N}^3$ satisfies the following conditions:

$$\begin{cases} \gcd(m, n) = \gcd(m, r-1) = 1, \\ r^n \equiv 1 \pmod{m} \end{cases} \tag{9}$$

Example 4.1 (I.C.P.,M.T.). We restricted the problem, by assuming that m prime and that n is the order of r modulo m.
If we assume that $Z(m, n, r)$ is quasi-Leinster, it follows that n is perfect and $m = n - 1$. This context overlaps with equation (2). Its four solutions $n = P_i$ provide examples of quasi-Leinster groups for all $r \in \{3, \ldots, n-1\}$.
If we take $r = 3$, we get the following examples:

- $(m, n, r) = (7, 6, 3)$,
- $(m, n, r) = (29, 28, 3)$,
- $(m, n, r) = (33550337, 33550336, 3)$,
- $(m, n, r) = (137438691329, 137438691328, 3)$.

Given $q = p^k$ with p prime and $k \in \mathbb{N}^*$, we consider the affine group

$$\mathrm{Aff}(\mathbb{F}_q) = \mathbb{F}_q \rtimes \mathbb{F}_q^* = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_q^*, b \in \mathbb{F}_q \right\}.$$

Proposition 4.3 (I.C.P.,M.T.).

- *There are no Leinster affine groups.*
- *There is just one almost Leinster affine group: for $q = 2$. This is isomorphic to C_2.*
- *There exist quasi-Leinster affine groups for $q - 1$ perfect.*

Example 4.2. Quasi-Leinster affine groups are given by solutions of (2): $\mathrm{Aff}(\mathbb{F}_7)$, $\mathrm{Aff}(\mathbb{F}_{29})$, $\mathrm{Aff}(\mathbb{F}_{33550337})$, and $\mathrm{Aff}(\mathbb{F}_{137438691329})$.

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