Tackling Number Theory analogues in Group Theory with GAP

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Group Functions

Let G be a finite group. We recall the following functions:

$$\sigma(G) = \sum_{N \lhd G} |N| \qquad H(G) = |G| \frac{\tau(G)}{\sigma(G)}$$

$$\rho(G) = \frac{\sigma(G)}{|G|} \qquad m(G) = \sum_{a \in G} \frac{1}{o(a)}$$

$$\tau(G) = \sum_{N \lhd G} 1$$

We introduce a new function:

$$h_m(G) = \frac{|G|}{m(G)}$$

Questions:

What groups G have $h_m(G) \in \mathbb{N}$? *Analogue of harmonic numbers*

What groups G check $\sigma(G) = 2|G| + 1$?

Analogue of quasi-perfect numbers: ***quasi - Leinster groups***

Integer harmonic mean of element orders

Theorem (I.C.P.,M.T.)

Let G be a finite p-group. Then $h_m(G) \in \mathbb{N}$ iff G is cyclic of order $p^{\sum_{i=1}^{s} p^i}$, with $s \in \mathbb{N}^*$ or $G \simeq D_8$.

Corollary (I.C.P.,M.T.)

The dihedral group D_8 is the only non-cyclic p-group with $h_m(G) \in \mathbb{N}$.

Proposition (I.C.P.,M.T.)

D_8 is the only dihedral group G with $h_m(G) \in \mathbb{N}$.

Theorem (I.C.P.,M.T.)

Let G be a finite group. Then $h_m(G) = 2$ iff $G \cong C_4$ or $G \cong D_8$.

Theorem (I.C.P.,M.T.)

There are no finite groups of odd order G with $h_m(G) = 3$.

Question (Open)

What is $\mathbb{N} \cap \mathrm{im}(h_m)$?

Conjecture

There are no groups G with $h_m(G) = 7$.

Tested with GAP for Small Groups.

A nice result

Proposition (Euclid)

Even perfect numbers are of the form $f(r) = 2^{r-1}(2^r - 1)$, where $r \geq 2$ and $2^r - 1$ is a Mersenne prime.

Example (The Great Internet Mersenne Prime Search)

There are 52 known perfect even numbers: $P_i, \quad i \in \{1, \dots, 52\}$.
The first seven are: $P_1 = 6 = f(2)$, $P_2 = 28 = f(3)$, $P_3 = 496 = f(5)$, $P_4 = 8128 = f(6)$, $P_5 = 33550336 = f(13)$, $P_6 = 8589869056 = f(17)$ and $P_7 = 137438691328 = f(19)$.

Corollary (I.C.P., M.T.)

Let p be a prime, P_i an even perfect number and k a positive integer such that:

$$P_i + 1 = p^k. \tag{1}$$

Then $k = 1$.

Example

Using GAP, we checked the first 39 even perfect numbers and found 4 solutions for (1):

$$P_i \in \{P_1 = 6, P_2 = 28, P_5 = 33550336, P_7 = 137438691328\}.$$

Quasi-Leinster groups

A **ZM-group** (Zassenhaus metacyclic) is a finite group with all Sylow subgroups cyclic.

$$ZM(m, n, r) = \langle a, b \mid a^m = b^n = 1, b^{-1}ab = a^r \rangle,$$

where the triplet $(m, n, r) \in \mathbb{N}^3$ satisfies the following conditions:

$$\begin{cases} \gcd(m, n) = \gcd(m, r-1) = 1, \\ r^n \equiv 1 \pmod{m} \end{cases} \tag{2}$$

We restricted the problem, by assuming that m prime and that n is the order of r modulo m.
If we assume that $Z(m, n, r)$ is quasi-Leinster, it follows that n is perfect and $m = n - 1$. This context overlaps with equation (1).
Its four solutions $n = P_i$ provide examples of quasi-Leinster groups for all $r \in \{3, \dots, n-1\}$.
If we take $r = 3$, we get the following examples:

- $(m, n, r) = (7, 6, 3)$,
- $(m, n, r) = (29, 28, 3)$,
- $(m, n, r) = (33550337, 33550336, 3)$,
- $(m, n, r) = (137438691329, 137438691328, 3)$.

References

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