The Ultimate Challenge: Rethinking The 3x+1 Problem

Rational Numbers as Terms of Periodic Generalized Collatz' Sequence

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ABSTRACT

Nicknamed as The Ultimate Challenge [1], The 3x+1 Problem is one of the longstanding problems of Computer Science and Mathematics. In essence, it says that whatever initial positive integer x_0 you choose, the recursive sequence $x_{n+1} = 3x_n + 1$ if x_n is an odd number and $x_{n+1} = \frac{x_n}{2}$ if x_n is an even number, will return $x_N = 1$ for some positive integer N. Even in its simplified form, which says that $1, 4, 2, 1, 4, 2, \ldots$ is the only periodic sequence that this recursive formula can produce, the problem is still unsolved. This easily stated problem shows how complex the behavior of a simple algorithm can be. In the earlier works [1,2] it was mentioned that the problem involves chaotic behavior and there are also undecidability features. In our study we prove that there is also a certain kind of order and pattern in the behavior of this algorithm. See [4] and the references therein for more insight.

INTRODUCTION

Consider the set of operations $F = \left\{T(x) = \frac{x}{2};\ S(x) = \frac{3x+1}{2};\ V(x) = \frac{3x+2}{2};\ W(x) = \frac{3x}{2}\right\}$. Suppose that a finite sequence of operations $P = B_0B_1 \ldots B_{n+m-1}$ is given, where $B_i \in F$ for $i = 0, 1, 2, \ldots, n+m-1$ and repetitions are allowed. For the sake of simplicity, we assume that $B_{n+m} = B_0$ and in general $B_i = B_j$ if $i \equiv j \pmod{n+m}$. So, P can be extended to be a sequence infinite both to the right and to the left. Suppose also that n is the number of T operations in finite P and m is the number of non-Toperations in finite P, i.e. the number of S, V and W operations together. We assume that m is positive and n is nonnegative. Consider the equation $B_0B_1 \ldots B_{n+m-2}B_{n+m-1}(x) = x$. It is a linear equation of x and its solution x_0 is a rational number. Similarly, denote the solution of $B_1 \ldots B_{n+m-2}B_{n+m-1}B_0(x) = x$ by x_1. Note that $x_0 = B_0(x_1)$. Next, denote the solution of $B_2 \ldots B_{n+m-2}B_{n+m-1}B_0B_1(x) = x$ by x_2. Note again that $x_1 = B_1(x_2)$. By continuing in this manner we determine other numbers x_i until we reach x_{n+m-1} which is the solution of $B_{n+m-1}B_0B_1 \ldots B_{n+m-2}(x) = x$ for which we also have $x_{n+m-1} = B_{n+m-1}(x_0)$. For completeness, we assume that $x_{n+m} = x_0$ and in general $x_i = x_j$ if $i \equiv j \pmod{n+m}$. We will prove that these solutions x_i, where $i = 0, 1, \ldots, n+m$, have some interesting properties. For this we need to introduce the numbers $U_i = \frac{2^i}{2^{n+m}-3^m}$, where $i = 0, 1, 2, \ldots, n+m$. The first number $U_0 = \frac{1}{2^{n+m}-3^m}$ can be interpreted as a solution of $U_0 = \frac{3^mU_0+1}{2^{n+m}}$. There is a pair of nonnegative integers (k, j), for which $3^kU_0 \pm U_j$ is an integer, or equivalently, $3^kU_i \pm U_{i+j}$ is an integer ($0 \le i \le i + j \le n+m$). For example, one can take trivial pairs $(k, j) = (0, 0)$ or other trivial pairs like $(k,\ j) = (m,\ n+m)$. But usually, we also have nontrivial pairs and if there is one then there are infinitely many of them. We are especially interested with the pairs (k, j) for which $k > 0$ and k is minimal, in the sense that there is no pair (k, j) with a smaller $k > 0$. We will call such pairs *primitive* and it is possible to prove that other pairs can be generated using these pairs. Note that for this and for the other pairs (k, j) satisfying the above condition we can also say that $2^{n+m} - 3^m$ divides $3^k \pm 2^j$. Similarly, there is a pair of non-negative integers (k_1, j_1), for which $3^{k_1}U_{j_1} \pm U_0$ is an integer, or equivalently, $3^{k_1}U_{i+j_1} \pm U_i$ is an integer ($0 \le i \le i + j_1 \le n+m$). Again, one can take trivial pairs like $(k_1, j_1) = (0, 0)$ and but there are also infinitely many other pairs. Again, for the pair (k_1, j_1) we can also say that $2^{n+m} - 3^m$ divides $3^{k_1}2^{j_1} \pm 1$. Finally, suppose that $\sigma(s, r)$, where $s \le r$, is the number of non-T operations in the fragment $B_sB_{s+2} \ldots B_{r-1}$ of the infinite sequence P. For concrete examples, one can experiment with [5].

MAIN RESULT

Theorem 1. For the pair of nonnegative integers (k, j) and for the numbers x_i defined above, $3^kx_i \pm 3^{\sigma(i,i+j)}x_{i+j}$ is an integer.

Theorem 2. For the pair of nonnegative integers (k_1, j_1) and for the numbers x_i, the number $3^{k_1+\sigma(i,i+j_1)}x_{i+j_1} \pm x_i$ is an integer.

Note: These results show that there are invariants, determined by the numbers U_i, which are independent of the order of operations in P. The proofs of the above results are based on the finite version of the Method of Mathematical Induction. The general setting of the problem for rational numbers looks as a promising direction.

REFERENCES

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[5] https://rextester.com/GRCO10408